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HW 2

MATH 563
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1. Koralov and Sinai, 2.1

   Find the probability that there are exactly 3 heads after five tosses of a fair coin.

   Solution. The number of possible outcomes of 5 tosses of a coin is $2^5 = 32$. The number of outcomes where there are 3 heads (or tails, for that matter) is $\binom{5}{3}$. Therefore the probability that there are exactly 3 heads after 5 tosses of a fair coin is

$$\begin{aligned} \text{P[3 heads in 5 tosses]} &= \frac{\binom{5}{3}}{32} \\ &= \frac{10}{32} \\ &= 0.3125. \end{aligned}$$

□

2. Koralov and Sinai, 2.2

   Andrew and Bob are playing a game of table tennis. The game ends when the first player reaches 11 points if the other player has 9 points or less. However, if at any time the score is 10:10, then the game continues till one of the players is 2 points ahead. The probability that Andrew wins any given point is 60 percent (it's independent of what happened before/during the game). What is the probability that Andrew will go on to win the game if he is currently ahead 9:8?

   Solution. First, consider the easiest outcomes (and most probable): when Andrew wins 11:9 or 11:8. The probability of these events (since probability of scoring a point is independent of what has happened previously in the game) is

$$\begin{aligned} \text{P[A wins 11:9 or A wins 11:8]} &= \text{P[A wins 11:9]} + \text{P[A wins 11:8]} \\ &= (0.6)^2 + \binom{3}{2}(0.6)^2(0.4)^1 \\ &= 0.7920 \end{aligned}$$

   However improbable there is the possibility that Bob will score at least two points, forcing Andrew to score more than 11 points total to win. The probability of these events can be represented with the following sum:

$$\text{P[A wins, but B scores 2 or more points]} = \sum_{m=2}^{\infty} \binom{2(m+1)}{m+1} (0.6)^{m+1}(0.4)^m$$

   where m represents the number of points Bob scores. It is clear that for large m, the contribution to the sum will be very small. The total probability, then, that Andrew wins given a 9:8 starting point is

$$P[\text{A wins}] \quad = \quad 0.7920 + \sum_{m=2}^{\infty} \binom{2(m+1)}{m+1} (0.6)^{m+1} (0.4)^m.$$

□

3. Koralov and Sinai, 2.3

Will you consider a coin asymmetric if after 1000 coin tosses the number of heads is equal to 600?

Solution. Consider the expectation and variance of sequence of n independent Bernoulli trials

$$\begin{aligned} \mathrm{E}[\nu^n] &= np \\ \mathrm{Var}[\nu^n] &= np(1-p). \end{aligned}$$

The Chebyshev Inequality gives us the relation

$$P[|\nu^n - \mathrm{E}[\nu^n]| \geq t] \leq \frac{\mathrm{Var}[\nu^n]}{t^2}.$$

For this problem, the desired deviation from the mean is $t = 100$. With $n = 1000$ trials $p = 0.5$, the above inequality becomes

$$\begin{aligned} P[|\nu^{1000} - \mathrm{E}[\nu^{1000}]| \geq 100] &\leq \frac{1000(0.5)(0.5)}{100^2} \\ &= \frac{250}{10000} \\ &= \frac{1}{40}. \end{aligned}$$

So, the probability of observing any sequence of trials with greater than 100 observations away from the mean is less than 0.03. This probability is quite low, and represents a large number of possible outcomes. Therefore, I would conclude that this coin is not symmetric. □

4. Koralov and Sinai, 2.4

Let ϵ_n be a numeric sequence such that $\epsilon_n \sqrt{n} \to +\infty$ as $n \to \infty$. Show that for a sequence of Bernoulli trials we have

$$P\Big[\Big|\frac{\nu^n}{n} - p\Big| < \epsilon_n\Big] \to 1 \text{ as } n \to \infty.$$

Solution. Consider the complementary event

$$P\Big[\Big|\frac{\nu^n}{n} - p\Big| \geq \epsilon_n\Big] \quad = \quad P[|\nu^n - np| \geq n\epsilon_n].$$

Now apply Chebyshev's Inequality (and use Theorem 2.4 from Koralov and Sinai) to get the result,

$$\begin{aligned}
\mathrm{P}[|\nu^n - np| \geq n\epsilon_n] &\leq \frac{\mathrm{Var}[\nu^n]}{n^2\epsilon_n^2} \\
&= \frac{np(1-p)}{n^2\epsilon_n^2} \\
&= \frac{p(1-p)}{n\epsilon_n^2} \\
&= \frac{p(1-p)}{(\sqrt{n}\epsilon_n)^2} \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

This proves the desired result, i.e. that

$$\mathrm{P}\Big[\Big|\frac{\nu^n}{n} - p\Big| < \epsilon_n\Big] \to 1 \text{ as } n \to \infty.$$

□

5. Koralov and Sinai, 2.6*

Let Ω be the space of sequences $\omega = (\omega_1, \ldots, \omega_n)$, where $\omega_i \in [0,1]$. Let P_n be the probability distribution corresponding to the homogeneous sequence of independent trials, each ω_i having uniform distribution on $[0,1]$. Let $\eta_n = \min_{1\leq i\leq n} \omega_i$. Find $\mathrm{P}_n[\eta_n \leq t]$ and $\lim_{n\to\infty} \mathrm{P}_n[n\eta_n \leq t]$.

Solution. Start with the calculation of $\mathrm{P}_n[\eta_n \leq t]$.

$$\begin{aligned}
\mathrm{P}_n[\eta_n \leq t] &= P\Big[\big\{\bigcap_{i=1}^n \{\omega_i > t\}\big\}^c\Big] \\
&= 1 - P\Big[\big\{\bigcap_{i=1}^n \{\omega_i > t\}\big\}\Big] \\
&= 1 - \prod_{i=1}^n \mathrm{P}[\omega_i > t] \text{ because the RVs are independent} \\
&= 1 - \prod_{i=1}^n \big(1 - \mathrm{P}[\omega_i \leq t]\big).
\end{aligned}$$

Now, define the distribution function for uniformly distributed RVs on the interval $[0,1]$,

$$\begin{aligned}
F(t) &= \mathrm{P}[\xi \leq t] \\
&= \begin{cases} 0 & t < 0 \\ t & t \in [0,1] \\ 1 & t > 1 \end{cases}.
\end{aligned}$$

We can rewrite the probability from above in terms of this distribution function,

$$\begin{aligned}
\mathrm{P}_n[\eta_n \le t] &= 1 - \prod_{i=1}^{n}\big(1 - \mathrm{P}[\omega_i \le t]\big) \\
&= 1 - \prod_{i=1}^{n}\big(1 - F(t)\big) \\
&= 1 - \big(1 - F(t)\big)^n \\
&= \begin{cases} 0 & t < 0 \\ 1 - \big(1 - t\big)^n & t \in [0,1] \\ 1 & t > 1. \end{cases}
\end{aligned}$$

Now, follow all of the steps above replacing t with t/n and we find that

$$\begin{aligned}
\mathrm{P}_n[n\eta_n \le t] &= \mathrm{P}_n[\eta_n \le t/n] \\
&= \begin{cases} 0 & t < 0 \\ 1 - \big(1 - \frac{t}{n}\big)^n & t \in [0,n] \\ 1 & t > n. \end{cases}
\end{aligned}$$

Take the limit as $n \to \infty$ and we obtain the result

$$\lim_{n\to\infty} \mathrm{P}_n[n\eta_n \le t] = \begin{cases} 0 & t < 0 \\ 1 - e^{-t} & t \in [0,\infty) \end{cases}$$

□

6. Koralov and Sinai, 2.8*

Consider a sequence of Bernoulli trials on a state space $X = \{0, 1\}$ with $p_0 = p_1 = 1/2$. Let $n \ge r \ge 1$ be integers. Find the probability that within the first n trials there appeared a sequence of r consecutive 1's.

Solution. Let $\nu^n = i$, $i < r$, corresponds to the scenarios where their cannot be r consecutive 1's. For $\nu^n = i$, $r \le i < n$, there could potentially be sequences of r consecutive 1's. Since $p = 1 - p$, we'll only be concerned with the cases $\nu^n = i > r$, as the number of total outcomes is 2^n.

For $\nu^n = r$, there are $n - (r - 1)$ ways that a sequence of r consecutive 1's can be observed.

For $\nu^n = r + 1$, there are again $n - (r - 1)$ ways that a sequence of r 1's can pop up. However, we need to avoid double counting. To do this we will not allow the last 1 to be placed at the beginning or the end of the sequence of r 1's. There are $n - (r + 1)$ remaining places for the additional 1 to appear. So, there is a total of $(n - (r - 1))(n - (r + 1))$ ways to have a sequence of at least r 1's.

Follow the argument inductively for $\nu^n = r+2, \ldots, n-1$, using a counting argument for the number of allowable combinations that don't double count sequences, we have the sum

$$\mathrm{P}[\nu^n = i] = \frac{(n-(r-1))\sum_{i=0}^{n-(r+1)} \binom{n-(r+1)}{i}}{2^n}$$